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The free energy of a collapsing branched polymer

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Abstract. We consider a number of related lattice models of branched polymers in dilute solution in which the polymer is modelled as a tree or as an animal. In order to model the effect on the thermodynamic properties of changing the temperature, or the quality of the solvent, we consider counting cycles in animals and near-neighbour contacts in both animals and trees. We show that the free energies of these models have common features and derive rigorous upper and lower bounds on the temperature dependence of the free energies. Finally, we derive series data for several of these models and compare our estimates of the limiting free energy with the rigorous bounds.

1. Introduction

A linear polymer molecule in dilute solution in a good solvent can be modelled as a self-avoiding walk on a regular lattice. If near-neighbour interactions are suitably weighted the (infinite) walk is thought to undergo a transition which models the internal transition in a polymer brought about by the dominance of attractive forces between monomers at low temperatures. This transition has been studied theoretically for many years (see e.g. Mazur and McCrackin 1968, Finsy *et al* 1975, Massih and Moore 1975, Morita 1976, Moore 1977, Post and Zimm 1979, Ishinabe 1985, Privman 1986, Chang *et al* 1988, Meirovitch and Lim 1989 and many other papers).

Randomly branched polymers in dilute solution in a good solvent can be modelled as lattice animals (i.e. as connected subgraphs of a lattice). Work by Lubensky and Isaacson (1979) suggested that cycles are relatively unimportant (in determining the universality class) and lattice trees have also been considered as a useful model of branched polymers in dilute solution.

As the solvent quality decreases, branched polymers are expected to become more compact and a collapse transition, analogous to that in linear polymers, is expected to occur. Numerical evidence for this transition has been provided by several groups (Derrida and Herrmann 1983, Dickman and Shieve 1984, 1986, Lam 1987, 1988, Chang and Shapir 1988). The models studied by these authors are formulated using different language but can all be expressed in terms of a cycle fugacity as the driving force for collapse (Gaunt and Flesia 1990). Dhar (1987) proved the existence of a collapse transition in a related model for directed animals.

A model closer to that used in studies of the internal transition in linear polymers is one in which interactions are introduced between pairs of vertices in the animal (or tree) which are neighouring sites of the lattice but which are not joined by an edge of the animal. In this paper our aim is to discuss a number of related models in which the collapse is driven by either a near-neighbour or cycle fugacity. In section 2 we discuss a variety of these models and then, in section 3, we prove some theorems which apply to all of them, showing that there are similarities in the behaviour of the limiting free energy as the appropriate fugacity varies. In section 4 we add some rigorous results for the compact phases of these models and, in section 5, we compare our results with predictions from analysis of exact enumeration data for several models in two and three dimensions. Finally in section 6 we discuss some possible extensions and limitations of this approach.

2. Models of collapsing branched polymers

Two basic types of model can be used for the collapse of branched polymers. In the first of these the collapse is driven by a near-neighbour contact fugacity, and in the second by a cycle fugacity. However, many variants of these two models are possible. To understand this we need some definitions and notation. We consider the d-dimensional hypercubic lattice, whose vertices are the integer points in R^d. We call these vertices lattice *sites*. If two lattice sites are unit distance apart they are joined by an *edge* of the lattice. There are two different types of lattice animal. A *site animal* is a connected section graph of the lattice (so that if two vertices of the animal are on adjacent lattice sites they must be connected by an edge in the animal). We shall write A_n for the number of site animals with n vertices where two animals are considered distinct if they cannot be superimposed by translation. For example, on the square lattice $A_1 = 1, A_2 = 2, A_3 = 6, A_4 = 19$, etc. A *bond animal* is a connected subgraph of the lattice (so that two vertices of the animal which are on adjacent lattice sites may or may not be connected by an edge in the animal). We write a_n for the number of distinct bond animals with n vertices where, once again, two animals are considered distinct if they cannot be superimposed by translation. On the square lattice $a_n = A_n$ for $n \leq 3$ and $a_4 = 23$.

Either site or bond animals can also be classified by the number of edges (rather than vertices) in the animal. We write B_n for the number of site animals with n edges and b_n for the number of bond animals with n edges. On the square lattice $B_1 = b_1 = 2$, $B_2 = b_2 = 6$, $B_3 = 18$, $b_3 = 22$, $B_4 = 56$ and $b_4 = 88$.

Yet another set of models is associated with lattice trees. A *site tree* is a site animal with no cycles and a *bond tree* is a bond animal with no cycles. We write T_n and t_n for the numbers of site trees and bond trees with n vertices. (Since the number of edges is just $n-1$ there is no need to consider separately the possibility of counting by edge content.) Again, for the square lattice, $T_n = t_n = A_n = a_n$ for $n \leq 3$, $T_4 = 18$ and $t_4 = 22$.

Some rigorous results are available about the asymptotic behaviour of the numbers a_n, etc. Using concatenation arguments it is easy to prove the existence of the following limits:

$$\lim_{n\to\infty} n^{-1} \log a_n = \log \lambda_s \qquad (2.1)$$

$$\lim_{n\to\infty} n^{-1}\log b_n = \log\lambda_b \tag{2.2}$$

$$\lim_{n\to\infty} n^{-1}\log A_n = \log\Lambda_s \tag{2.3}$$

$$\lim_{n\to\infty} n^{-1}\log B_n = \log\Lambda_b \tag{2.4}$$

$$\lim_{n\to\infty} n^{-1}\log t_n = \log\lambda_0 \tag{2.5}$$

$$\lim_{n\to\infty} n^{-1}\log T_n = \log\Lambda_0 \tag{2.6}$$

where λ_s, etc are called *growth constants*. These growth constants are known to be finite (Klarner 1967, Soteros and Whittington 1990).

We now return to constructing models for collapse. If two vertices of an animal are adjacent on the lattice but are not incident on the same edge in the animal we say that these two vertices form a *contact*, or that they are non-bonded near-neighbours. For site animals and for site trees the number of contacts is zero, so contact models can only be constructed for bond animals and for bond trees. We write $a_n(u), b_n(u), t_n(u)$ for the number of bond animals with n vertices and u contacts, bond animals with n edges and u contacts, and bond trees with n vertices and u contacts. For example, on the square lattice $a_4(1) = b_3(1) = t_4(1) = 4$. The corresponding partition functions will be written as

$$Z_n(\beta; a) = \sum_u a_n(u)\mathrm{e}^{\beta u} \tag{2.7}$$

etc, and the corresponding reduced limiting free energies as

$$G(\beta; a) = \lim_{n\to\infty} n^{-1}\log Z_n(\beta; a). \tag{2.8}$$

To obtain corresponding expressions for the other models we replace a by b, or t.

If we weight animals (etc) by their cyclomatic index (i.e. the maximum number of edges which can be removed without disconnecting the animal) then we are restricted to bond and site animals (since trees have cyclomatic index zero). We write $a_n^o(c)$, $b_n^o(c)$, $A_n^o(c)$, and $B_n^o(c)$ for the number of bond animals with n vertices, bond animals with n edges, site animals with n vertices and site animals with n edges, each having cyclomatic index c. The superscript o indicates the classification by cycles. The corresponding partition functions and limiting free energies will be written as

$$Z_n^o(\beta; a) = \sum_c a_n^o(c)\mathrm{e}^{\beta c} \tag{2.9}$$

and

$$G^o(\beta; a) = \lim_{n\to\infty} n^{-1}\log Z_n^o(\beta; a) \tag{2.10}$$

etc, where a can be replaced by b, A or B as appropriate.

3. Some general theorems

The models described in section 2 have many features in common and, in this section, we emphasise this by proving some general results which apply to all of the models.

In particular, we prove the existence of a limiting free energy, show that this function is convex, monotone and continuous, and derive some bounds on the function.

We write $f_n(c)$ for the number of embeddings of a certain type of graph, where n will be the number of vertices or edges in the graph and c will be the number of contacts or cycles. Consequently $f_n(c)$ is a non-negative integer defined for all $n > 0$ and $c \geq 0$. We define

$$F_n(\beta) = \sum_{c=0}^{\infty} f_n(c) e^{\beta c} \tag{3.1}$$

$$f_n = F_n(0) = \sum_c f_n(c) \tag{3.2}$$

$$F_n(-\infty) \equiv f_n(0) \tag{3.3}$$

and

$$c_{\max}(n) = \max\{c | f_n(c) \neq 0\}. \tag{3.4}$$

We shall be interested in functions f_n and $c_{\max}(n)$ satisfying

$$\limsup_{n\to\infty} n^{-1} \log f_n < \infty \tag{3.5}$$

and

$$\limsup_{n\to\infty} n^{-1} c_{\max}(n) < \infty. \tag{3.6}$$

Since we expect to be able to concatenate embeddings in pairs in such a way that contacts or cycles are additive, $f_n(c)$ will satisfy

$$f_{n+m}(c) \geq \sum_{i=0}^{c} f_n(i) f_m(c-i) \tag{3.7}$$

for all $m > 0$, $n > 0$ and $c \geq 0$.

Given suitable functions satisfying equations (3.1) to (3.7) we now prove a series of lemmas.

Lemma 3.1. $\lim_{n\to\infty} n^{-1} \log F_n(\beta) = \mathcal{F}(\beta)$ exists for $-\infty \leq \beta < \infty$, and $\mathcal{F}(\beta)$ is monotone non-decreasing, convex and continuous for $-\infty < \beta < \infty$.

Proof. Multiplying (3.7) by $e^{c\beta}$ and summing over c gives

$$F_{m+n}(\beta) \geq F_m(\beta) F_n(\beta). \tag{3.8}$$

Then (3.5), (3.6) and (3.8) imply that $\lim_{n\to\infty} n^{-1} \log F_n(\beta)$ exists and is finite, and that

$$\sup_{n>0} n^{-1} \log F_n(\beta) = \lim_{n\to\infty} n^{-1} \log F_n(\beta) = \mathcal{F}(\beta) \tag{3.9}$$

for $-\infty \leq \beta < \infty$. Since $F_n(\beta)$ is monotone increasing, it follows that $\mathcal{F}(\beta)$ is monotone non-decreasing. Therefore, to prove that $\mathcal{F}(\beta)$ is convex (and hence continuous), it suffices to prove that

$$\tfrac{1}{2}(\mathcal{F}(\beta_1) + \mathcal{F}(\beta_2)) \geq \mathcal{F}\left(\frac{\beta_1 + \beta_2}{2}\right) \tag{3.10}$$

and (3.10) follows immediately from

$$\begin{aligned} F_n(\beta_1)F_n(\beta_2) &= \sum_{c_1} f_n(c_1)\mathrm{e}^{\beta_1 c_1} \sum_{c_2} f_n(c_2)\mathrm{e}^{\beta_2 c_2} \\ &\geq \left(\sum_c f_n(c)\mathrm{e}^{(\beta_1+\beta_2)c/2}\right)^2 \\ &= \left[F_n\left(\frac{\beta_1+\beta_2}{2}\right)\right]^2 \end{aligned} \tag{3.11}$$

on taking logarithms, dividing by n and letting $n \to \infty$.

Lemma 3.2. $\lim_{n\to\infty} c_{\max}(n)/n$ exists and is equal to $\sup_{n>0} c_{\max}(n)/n \equiv M$.

Proof. From (3.7)

$$f_{n+m}(c_{\max}(n) + c_{\max}(m)) \geq f_n(c_{\max}(n))f_m(c_{\max}(m)) > 0 \tag{3.12}$$

so that

$$c_{\max}(n+m) \geq c_{\max}(n) + c_{\max}(m) \tag{3.13}$$

and the lemma follows from (3.6) and (3.13).

Lemma 3.3. For $\beta > 0$

$$M\beta \leq \mathcal{F}(\beta) \leq \mathcal{F}(0) + M\beta \tag{3.14}$$

and

$$\lim_{\beta\to+\infty} \mathcal{F}(\beta)/\beta = M. \tag{3.15}$$

Moreover, there is an asymptotic line $L(\beta) = M\beta + S$ such that $\lim_{\beta\to\infty} \mathcal{F}(\beta) - L(\beta) = 0$.

Proof.

$$F_n(\beta) = \sum_c f_n(c)\mathrm{e}^{\beta c} \geq f_n(c_{\max}(n))\mathrm{e}^{\beta c_{\max}(n)} \tag{3.16}$$

for all β, so that

$$\mathcal{F}(\beta) \geq \limsup_{n\to\infty} n^{-1} \log f_n(c_{\max}(n)) + M\beta \geq M\beta. \tag{3.17}$$

For $\beta > 0$

$$F_n(\beta) \le \sum_c f_n(c) e^{\beta c_{\max}(n)} = f_n e^{\beta c_{\max}(n)} \tag{3.18}$$

so that

$$\mathcal{F}(\beta) \le \mathcal{F}(0) + M\beta. \tag{3.19}$$

(3.15) follows on dividing (3.14) by β and letting $\beta \to \infty$. $\mathcal{F}(\beta) - M\beta$ is convex and bounded between 0 and $\mathcal{F}(0)$ for all $\beta \ge 0$. Therefore $\mathcal{F}(\beta) - M\beta$ must be non-increasing in β, hence tending to a limit which we denote S. This implies the existence of an asymptotic line.

Remark. S will be interpreted as the reduced limiting entropy of the compact phase (see section 4).

Lemma 3.4. If $\forall\, \epsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{c=0}^{\delta n} f_n(c) \le f_n(0)(1+\epsilon)^n \qquad \forall n \tag{3.20}$$

then

$$\lim_{\beta \to -\infty} \mathcal{F}(\beta) = \mathcal{F}(-\infty). \tag{3.21}$$

Proof. For $\beta < 0$

$$f_n(0) \le F_n(\beta) \le \sum_{c=0}^{\delta n} f_n(c) + \sum_{c=\delta n}^{\infty} f_n(c) e^{\beta c} \tag{3.22}$$

so from (3.20)

$$f_n(0) \le F_n(\beta) \le f_n(0)(1+\epsilon)^n + e^{\beta\delta n} F_n(0). \tag{3.23}$$

Taking logarithms, dividing by n and letting $n \to \infty$ gives

$$\mathcal{F}(-\infty) \le \mathcal{F}(\beta) \le \max[\mathcal{F}(-\infty) + \log(1+\epsilon), \mathcal{F}(0) + \beta\delta]. \tag{3.24}$$

Now let $\beta \to -\infty$ so that

$$\mathcal{F}(-\infty) \le \liminf_{\beta \to -\infty} \mathcal{F}(\beta) \le \limsup_{\beta \to -\infty} \mathcal{F}(\beta) \le \mathcal{F}(-\infty) + \log(1+\epsilon) \tag{3.25}$$

and then let $\epsilon \to 0+$ which establishes (3.21).

We have the following corollary to lemma 3.4

Lemma 3.5. If, for some constants $A, B \ge 0$

$$f_n(c) \le \binom{An}{c} B^n f_n(0) \tag{3.26}$$

for all n and c, then (3.20) is satisfied and hence (3.21).

Proof. From (3.26) above and (3.21) of Madras *et al* (1988)

$$\sum_{c=0}^{\delta n} f_n(c) \le \sum_{c=0}^{(An)(\delta/A)} \binom{An}{c} B^c f_n(0)$$
$$\le \left(\frac{B^{\delta/A}}{(\delta/A)^{\delta/A}(1-\delta/A)^{1-\delta/A}}\right)^{An} f_n(0). \tag{3.27}$$

But

$$\lim_{\delta\to 0+} \left(\frac{B^{\delta/A}}{(\delta/A)^{\delta/A}(1-\delta/A)^{1-\delta/A}}\right)^{A} = 1 \tag{3.28}$$

so that (3.27) and (3.28) imply (3.20).

Remark. If $f_n(0)$ is replaced by $e^{n\mathcal{F}(-\infty)}$ in (3.20) or in (3.26) then the resulting conditions still imply (3.21).

We now consider the application of lemmas 3.1–3.5 to the seven models discussed in section 2. There are four models in which we weight by cyclomatic index (i.e. bond animals counted by vertices (model a^o), bond animals counted by edges (model b^o), site animals counted by vertices (model A^o) and site animals counted by edges (model B^o)), and three models in which we weight by the number of contacts (i.e. bond animals counted by vertices (model a), bond animals counted by edges (model b) and bond trees counted by vertices (model t)).

Since

$$\lim_{n\to\infty} n^{-1} \log a_n < \infty \tag{3.29}$$

(3.5) is satisfied for all seven models. Similarly, (3.6) is satisfied (by arguments analogous to lemma 2.1 in Madras *et al* (1988)). In each case the graphs can be concatenated in pairs so that contacts or cycles are additive, so that (3.7) is satisfied. (Note that when counting by bonds an extra bond must be added, and so we take $f_n(c) = b_{n-1}(c)$, etc) Hence, for all seven models, the conditions of lemmas 3.1, 3.2 and 3.3 are satisfied. To complete the picture (as far as these lemmas are concerned) we need to identify $\mathcal{F}(-\infty)$, $\mathcal{F}(0)$ and M for the seven models.

Model a^o: $\mathcal{F}(0) = \log \lambda_s$	$\mathcal{F}(-\infty) = \log \lambda_0$	$M = d-1$
Model b^o: $\mathcal{F}(0) = \log \lambda_b$	$\mathcal{F}(-\infty) = \log \lambda_0$	$M = (d-1)/d$
Model A^o: $\mathcal{F}(0) = \log \Lambda_s$	$\mathcal{F}(-\infty) = \log \Lambda_0$	$M = d-1$
Model B^o: $\mathcal{F}(0) = \log \Lambda_b$	$\mathcal{F}(-\infty) = \log \Lambda_0$	$M = (d-1)/d$
Model a: $\mathcal{F}(0) = \log \lambda_s$	$\mathcal{F}(-\infty) = \log \Lambda_s$	$M = d-1$
Model b: $\mathcal{F}(0) = \log \lambda_b$	$\mathcal{F}(-\infty) = \log \Lambda_b$	$M = d-1$
Model t: $\mathcal{F}(0) = \log \lambda_0$	$\mathcal{F}(-\infty) = \log \Lambda_0$	$M = d-1$.

Lemmas 3.4 and 3.5 are concerned with the limiting behaviour as $\beta \to -\infty$. The idea is to devise a construction to reduce the number of cycles or contacts. For bond

animals one can reduce the number of cycles by deleting edges or reduce the number of contacts by adding edges. From (2.10) of Madras *et al* (1988) it follows that

$$a_n^o(c) \leq \binom{(d-1)n}{c} a_n^o(0) \tag{3.30}$$

so that model a^o satisfies the conditions of lemma 3.5 and hence

$$\lim_{\beta \to -\infty} G^o(\beta; a) = \log \lambda_0. \tag{3.31}$$

The corresponding inequality when counting by edges (model b^o) is immediate:

$$\begin{aligned} b_n^0(c) &\leq \binom{(d-1)n}{c} b_{n-c}^0(0) \\ &\leq \binom{(d-1)n}{c} b_n^0(0) \end{aligned} \tag{3.32}$$

so that

$$\lim_{\beta \to -\infty} G^o(\beta; b) = \log \lambda_0. \tag{3.33}$$

In a similar way we can add edges to bond animals to reduce the number of contacts. For any bond animal with c contacts we can add c additional edges to reduce the number of contacts to zero but a bond animal with no contacts can have more than one precursor with c contacts. Since the animal (with n vertices) can have no more than dn edges we have

$$a_n(c) \leq \binom{dn}{c} a_n(0) \tag{3.34}$$

so that

$$\lim_{\beta \to -\infty} G(\beta; a) = \log \Lambda_s. \tag{3.35}$$

Similarly

$$b_n(c) \leq \binom{n+c}{c} b_{n+c}(0) \tag{3.36}$$

for $c \leq (d-1)n$, so that

$$b_n(c) \leq \binom{dn}{c} e^{c\mathcal{F}(-\infty)} e^{n\mathcal{F}(-\infty)}. \tag{3.37}$$

Hence

$$\lim_{\beta \to -\infty} G(\beta; b) = \log \Lambda_b. \tag{3.38}$$

We do not have corresponding constructions for decreasing the number of contacts in bond trees or cycles in site animals.

The inequalities in (3.30), (3.32), (3.34) and (3.37) not only establish the behaviour in the $\beta \to -\infty$ limit but also give rise to upper bounds on the free energy which are useful for $\beta < 0$. For instance, from (3.30) we have

$$\begin{aligned} Z_n^o(\beta; a) &\leq \sum_c \binom{(d-1)n}{c} a_n^o(0) \mathrm{e}^{\beta c} \\ &= (1+\mathrm{e}^{\beta})^{(d-1)n} a_n^o(0) \end{aligned} \tag{3.39}$$

so that

$$G^o(\beta; a) \leq \log \lambda_0 + (d-1)\log(1+\mathrm{e}^{\beta}). \tag{3.40}$$

Similarly we obtain

$$G^o(\beta; b) \leq \log \lambda_0 + (d-1)\log(1+\mathrm{e}^{\beta}) \tag{3.41}$$

$$G(\beta; a) \leq \log \Lambda_s + d\log(1+\mathrm{e}^{\beta}) \tag{3.42}$$

and

$$G(\beta; b) \leq \log \Lambda_b + d\log(1+\Lambda_b \mathrm{e}^{\beta}). \tag{3.43}$$

For some cases it is possible to obtain improved bounds for $\beta > 0$ and in the next lemma we give an improved result for $G^o(\beta; a)$.

Lemma 3.6. For all β

$$\begin{aligned} (d-1)\beta + (d-1)\log(1+\mathrm{e}^{-\beta}) &\leq G^o(\beta; a) \\ &\leq (d-1)\beta + d\log\left(\frac{2\mathrm{e}^{\beta}}{1+2\mathrm{e}^{\beta}-\sqrt{4\mathrm{e}^{\beta}+1}}\right). \end{aligned} \tag{3.44}$$

Proof. From lemma 2.3 of Madras *et al* (1988)

$$\binom{c+k}{k} a_n^o(c+k) \leq \binom{c_{\max}(n)-c}{k} a_n^o(c). \tag{3.45}$$

To get the lower bound first replace $c+k$ by $c_{\max}(n)$ in (3.45) to give

$$\binom{c_{\max}(n)}{k} a_n^o(c_{\max}(n)) \leq a_n^o(c_{\max}(n)-k). \tag{3.46}$$

and then multiply by $\mathrm{e}^{\beta c}$ and sum over c to give

$$Z_n^o(\beta; a) \geq a_n^o(c_{\max}(n))(1+\mathrm{e}^{\beta})^{c_{\max}(n)}. \tag{3.47}$$

The lower bound follows by taking logarithms dividing by n and letting $n \to \infty$.

For the upper bound, let $a_{n,e,i}$ be the number of animals (up to translation) with n vertices, e edges and i boundary edges. Note that $i \equiv 2dn - 2e - l$ where l is the number of contacts in the animal. By arguing as in Kesten (1982, lemma 5.1) for the edge-percolation model, we have

$$\sum_{n=1}^{\infty}\sum_{e=0}^{\infty}\sum_{i=1}^{\infty} n a_{n,e,i} p^e (1-p)^i \leq 1 \tag{3.48}$$

for any p, $0 < p < 1$. The upper bound follows from this inequality by considering animals with c cycles and n vertices. Hence $e \equiv c+n-1$ and $2dn-2e+c-c_{\max}(n) \leq i \leq 2dn-2e$, so that

$$a_n^o(c) = \sum_{i=2dn-2e+c-c_{\max}(n)}^{2dn-2e} a_{n,e,i}. \tag{3.49}$$

The inequality (3.48) implies

$$\sum_{i=1}^{2dn-2e} n a_{n,e,i} p^e (1-p)^{2dn-2e} \leq 1. \tag{3.50}$$

Combining equation (3.49) and equation (3.50)

$$a_n^o(c) \leq \frac{1}{np^e(1-p)^{2dn-2e}} \tag{3.51}$$

where $e \equiv c+n-1$. Multiplying both sides by $e^{\beta c}$, summing over $0 \leq c \leq c_{\max}(n)$ and choosing p such that $p = e^{\beta}(1-p)^2$ gives

$$Z_n^o(\beta;a) \leq \frac{(c_{\max}(n)+1)e^{\beta((d-1)n+1)}}{np^{dn}}. \tag{3.52}$$

The upper bound for $G^o(\beta;a)$ follows by dividing equation (3.52) by n, taking logarithms, substituting $p = 1+(1-\sqrt{1+4e^{\beta}})/2e^{\beta}$ and letting $n \to \infty$.

Similarly for $G^o(\beta;b)$ the inequality (3.45) leads to a lower bound and the inequality (3.48) leads to an upper bound giving

$$\frac{(d-1)}{d}\beta + \frac{(d-1)}{d}\log(1+e^{-\beta}) \leq G^o(\beta;b) \leq \frac{(d-1)}{d}\beta - \log(1-e^{-\frac{\beta}{2d}}). \tag{3.53}$$

Using similar arguments one can derive useful bounds for model A^o and for model t. For model A^o the result comes from

$$\sum_{j=c}^{c_{\max}(n)} \binom{j}{j-c} A_n^o(j) \leq a_n^o(c). \tag{3.54}$$

Multiplying by $(e^{\beta}-1)^c$, summing from $c = 0, \ldots, c_{\max}(n)$, reversing the order of summation, taking logarithms, dividing by n and letting $n \to \infty$ gives for $\beta > 0$

$$G^o(\beta;A) \leq G^o(\beta+\log(1-e^{-\beta});a)$$
$$\leq (d-1)\beta+(d-1)\log(1-e^{-\beta})+d\log\left(\frac{2e^{\beta}-2}{2e^{\beta}-1-\sqrt{4e^{\beta}-3}}\right). \tag{3.55}$$

An alternative upper bound on $G^o(\beta;A)$ can be derived from the inequality analogous to (3.48) for site percolation. Then the appropriate perimeter is the *site perimeter* but this can never be less than the bond perimeter since each perimeter site must be connectable to the cluster by at least one perimeter bond. Using this one readily obtains

$$G^o(\beta;A) \leq (d-1)\beta - \log(1-\mathrm{e}^{-\beta/2}) \tag{3.56}$$

which is an improvement over the upper bound in (3.55) for large β. For model t the result follows from

$$t_n(k) \leq \binom{k+n-1}{k} A_n^o(k). \tag{3.57}$$

Multiplying by $\mathrm{e}^{\beta k}$, summing from $k=0,...,c_{\max}(n)$, taking logarithms, dividing by n and letting $n \to \infty$

$$G(\beta;t) \leq d\log d - (d-1)\log(d-1) + G^o(\beta,A). \tag{3.58}$$

4. The compact phase

In this section we discuss the limiting entropy of the compact phase for these models. This is defined as the intercept S of the asymptotic line $L(\beta)$ of the free energy $\mathcal{F}(\beta)$ (see lemma 3.3). In some cases we can calculate the value of the intercept directly and in others we shall be interested in the value of $\limsup_{n\to\infty} n^{-1}\log f_n(c_{\max}(n))$, which is a lower bound for S (see the left inequality in (3.17)). In particular, we are interested in whether or not this quantity is zero and, if not, on a lower bound for its value.

For the cycles models we have the following theorem.

Theorem 4.1 For all four cycles models (a^o, b^o, A^o, and B^o), the limiting entropy S is zero. Moreover, for the two bond animal models (a^o and b^o) the free energy $\mathcal{F}(\beta)$ is never equal to the asymptotic line $L(\beta)$ for any finite β.

Proof. For models a^o and b^o, both results follow from (3.44) and (3.53) respectively. Next, a limiting entropy of zero for a^o implies the same for A^o, because $G^o(\beta;a) \geq G^o(\beta;A) \geq (d-1)\beta$. Similarly, a limiting entropy of zero for b^o implies the same for B^o.

Theorem 4.1 implies that

$$\lim_{n\to\infty} n^{-1}\log a_n(c_{\max}(n)) = 0 \tag{4.1}$$

and similar results for b^o, A^o, and B^o. Essentially, this is because the animals with the maximum number of cycles are cubes in which every edge is present, and so the number of such animals does not grow exponentially. For contact models, however, the situation is quite different.

For the contact model for bond animals, the structure with the maximum number of contacts must be a tree (because otherwise we could remove an edge and increase the number of contacts). Moreover, the set of sites of such a maximally compact animal must (approximately) fill a cube. In the next theorem, we show that there are exponentially many such compact animals; that is, the limiting entropy is strictly positive.

Theorem 4.2. For each of the three contact models (a, b, and t), the limiting entropy S is greater than or equal to $4C/\pi = 1.166\ldots$, where C is Catalan's constant.

Proof. In d dimensions, consider a cube with m^d vertices and put $n = m^d$. Then $t_n(c_{\max}(n))$ is the number of spanning trees of this cube, and $a_n(c_{\max}(n)) = b_{n-1}(c_{\max}(n-1)) = t_n(c_{\max}(n))$. Therefore it will suffice to prove that

$$\lim_{n=m^d\to\infty} n^{-1}\log t_n(c_{\max}(n)) \geq 4C/\pi. \tag{4.2}$$

We first discuss the proof for $d = 2$. The $m \times m$ square is almost a regular graph of degree 4. It can be made regular by adding a loop to every vertex of degree 3, and two loops to each of the four vertices of degree 2, with the understanding that a loop is counted as an incident edge only once. Observe that the addition of loops does not change the number of spanning trees.

The number of spanning trees of the resulting graph H is

$$\tau(n) \equiv \frac{1}{n}\prod(4-\lambda) \tag{4.3}$$

where the product is over all of the eigenvalues λ of H except the one that equals 4. (See proposition 1.4 and the discussion that follows in Cvetković *et al* (1979).) The graph H is the *sum* of two paths with m vertices and with a (singly counted) loop at each end. These paths have eigenvalues $2\cos(\pi i/m)$, $i = 0, 1, \ldots, m-1$, so the eigenvalues of H are given by

$$\lambda_{ij} = 2\cos(\pi i/m) + 2\cos(\pi j/m) \tag{4.4}$$

where i and j each run from 0 to $m-1$. (See Cvetković *et al* (1979), section 2.5 for further details.) This gives

$$\lim_{n\to\infty} n^{-1}\log\tau(n) = \lim_{m\to\infty} m^{-2}\sum_{i,j}\log(4-2\cos(\pi i/m)-2\cos(\pi j/m)) \tag{4.5}$$

and, converting sums to integrals,

$$\lim_{n\to\infty} n^{-1}\log\tau(n) = \pi^{-2}\int_0^\pi\int_0^\pi \log(4-2\cos\alpha-2\cos\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{4.6}$$

The value of this integral is $4C/\pi$ (see for instance Kasteleyn (1961)). By using the same argument in d dimensions, (4.6) generalizes to

$$\lim_{n=m^d\to\infty} n^{-1}\log t_n(c_{\max}(n)) = \pi^{-d}\int_0^\pi\cdots\int_0^\pi \log(2d-2\sum_{i=1}^{d}\cos\alpha_i)\,d\alpha_1\ldots d\alpha_d. \tag{4.7}$$

This is increasing in d, which proves the theorem.

The following bound is an immediate consequence of (3.58):

Theorem 4.3. For the tree contact model, the limiting entropy S is at most $d\log d - (d-1)\log(d-1)$.

So, in particular, for the tree contact model in two dimensions, $1.166\ldots \leq S \leq 1.386\ldots$

5. Numerical results

In this section we report numerical estimates for the β dependence of the limiting free energy for several of these models. These estimates are based on exact enumeration data derived using a method (Sykes 1986 a, b, c, d, Martin 1990) related to the *shadow method* (Sykes *et al* 1965). The number of animals with n vertices, classified by edges and contacts, has been derived for $n \leq 19$ on the square lattice and for $n \leq 17$ on the simple cubic lattice. These data are given in appendices A and B.

For each of the models which we consider we assume that the appropriate partition function behaves as

$$F_n(\beta) \sim n^{-\theta(\beta)} e^{\mathcal{F}(\beta)n} \tag{5.1}$$

which is consistent with lemma 3.1. We write $r_n(\beta)$ for the ratio

$$r_n(\beta) = F_n(\beta)/F_{n-1}(\beta) \tag{5.2}$$

and (5.1) then implies that

$$r_n(\beta) = e^{\mathcal{F}(\beta)}(1 - \theta(\beta)/n + \dots) \tag{5.3}$$

so $\mathcal{F}(\beta)$ can be estimated by standard ratio techniques (Gaunt and Guttmann 1974).

We also construct the generating functions

$$\mathcal{G}(x) = 1 + \sum_{n\geq 1} F_n(\beta) x^n \tag{5.4}$$

and (5.1) then implies that $\mathcal{G}(x)$ behaves as

$$\mathcal{G}(x) \sim (1 - e^{\mathcal{F}(\beta)}x)^{\theta(\beta)-1} \tag{5.5}$$

close to $x = e^{-\mathcal{F}(\beta)}$. The limiting free energy $\mathcal{F}(\beta)$ can then be estimated from the location of the appropriate pole in the logarithmic derivative of $\mathcal{G}(\beta)$ by Padé approximant techniques (Gaunt and Guttmann 1974).

For each of the models which we have studied both of these methods work well for $\beta \leq 0$ and for small positive values of β. The estimates from the two methods agree well with each other but the ratio estimates are usually more precise. For larger values of β both methods become less useful and eventually fail to provide reliable estimates of $\mathcal{F}(\beta)$. This feature will be discussed in section 6.

In figures 1–3 we give our estimates of $\mathcal{F}(\beta)$ for models a^o, b^o and a for the square lattice. In each case we include the upper and lower bounds derived in sections 3 and 4, using numerical estimates of the values of $\mathcal{F}(-\infty)$ and $\mathcal{F}(0)$. For all three cases, the approach to $\mathcal{F}(-\infty)$ is very rapid as β becomes more negative. Similarly, for models a^o and b^o, the free energy rapidly approaches the asymptote as β becomes more positive.

In figures 4 and 5 we give similar estimates of $\mathcal{F}(\beta)$ for models a^o and b^o on the simple cubic lattice. The results are very satisfactory for $\beta \leq 0$ but the estimates rapidly become imprecise as β increases.

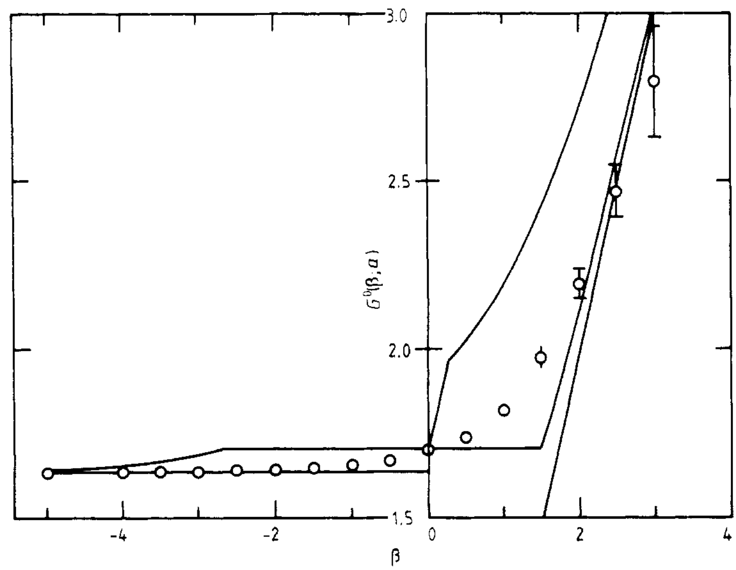

Figure 1. Numerical estimates of the reduced limiting free energy for the cycles model with site counting on the square lattice. Estimated errors are given unless smaller than the symbols. Upper and lower bounds, and asymptotes, are included.

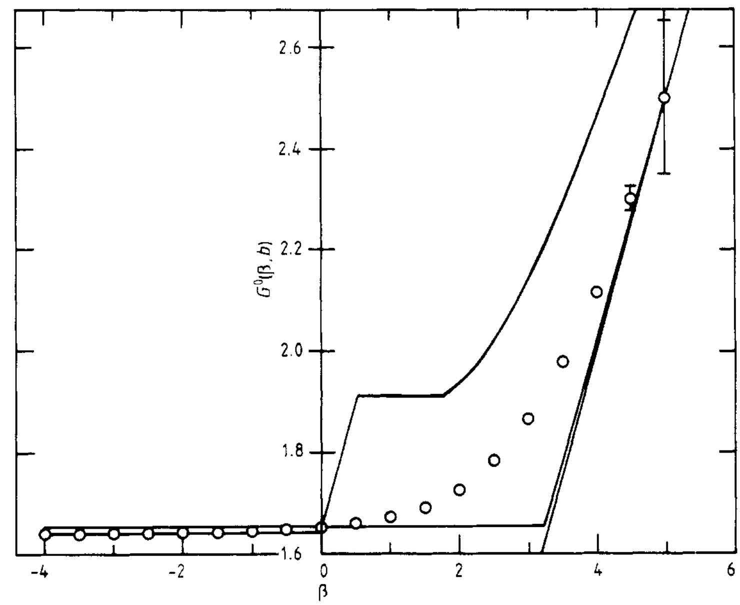

Figure 2. As figure 1 but for bond counting.

6. Discussion

We have considered a number of closely related lattice models of branched polymers in which the collapse of the polymer is driven either by a cycle fugacity or by a fugacity associated with near-neighbour contacts in the animal. In section 3 we derived some

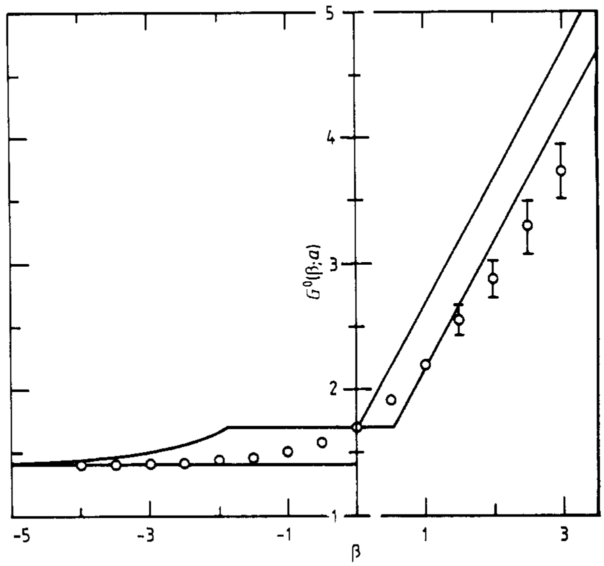

Figure 3. Numerical estimates of the reduced limiting free energy for the contact model with site counting on the square lattice. Upper and lower bounds are given as continuous curves.

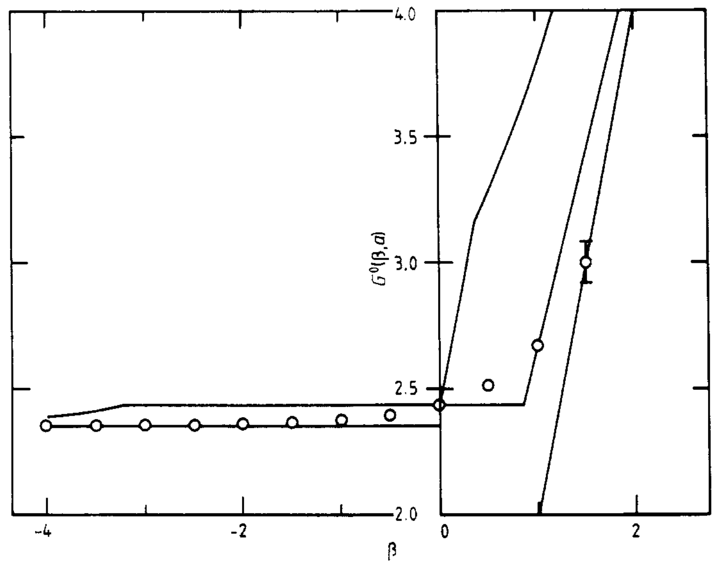

Figure 4. As figure 1 but on the simple cubic lattice.

general theorems showing that the limiting free energy exists for all of these models, and shares some common properties such as convexity and continuity. We have also derived some upper and lower bounds on the dependence of the free energy on β and, in some cases, these bounds allow us to establish the $\beta \to -\infty$ and $\beta \to \infty$ behaviour in some detail. We have also characterised the limiting entropy of the compact phase for some models and derived bounds on its value for others.

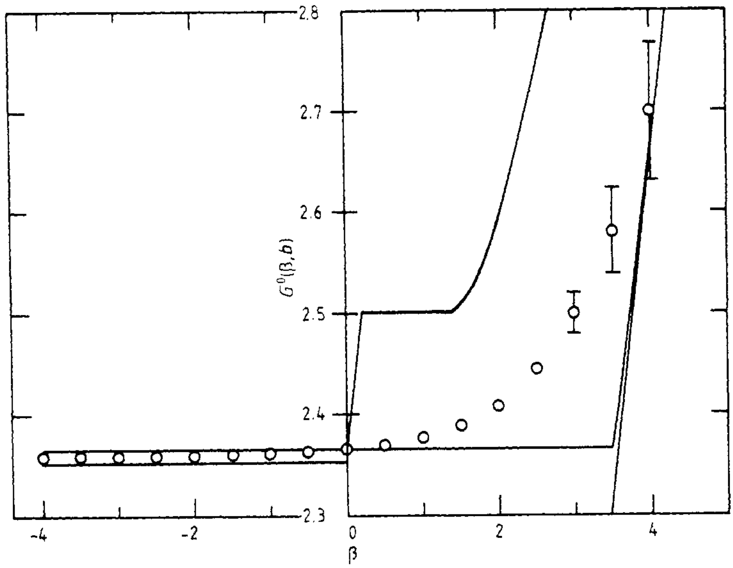

Figure 5. As figure 2 but on the simple cubic lattice.

Our analysis of exact enumeration data gives a more detailed picture which is consistent with the behaviour predicted above. We mentioned earlier that the series analysis methods worked well for $\beta \leq 0$ and for small positive values of β, but progressively less well as β increased. We believe that clusters of the size which we have been able to generate are not typical of clusters with a large number of cycles or contacts, because surface effects are still important at these values of n. This will be even more serious in three dimensions than in two and we find that the analysis techniques fail earlier in higher dimension.

The free energy $\mathcal{F}(\beta)$ is expected to be analytic except for some positive value of β corresponding to the collapse transition. We have been unable to establish this rigorously and, as expected, the estimated free energy curves derived from the exact enumeration data show no sign of this transition. One expects that it will be necessary to examine heat capacity data to locate the transition and some preliminary results on this have already been published (Gaunt and Flesia 1990).

Acknowledgments

We would like to thank Professor H N V Temperley for a suggestion that led to theorem 4.2. This work was partially supported by NSERC of Canada.

Appendix A. Exact enumeration data on the square lattice. The number of clusters with n vertices, e edges and u contacts is the coefficient of $x^n b^e \lambda^u$. The computation relies on combinatorial methods invented by Sykes (1986a, b, c, d) implemented as a general purpose package by Martin (1990). The University of London CRAY was used, and the results required a total of about 10 hours of CPU time. (It should be said that a direct count without the benefit of combinatorial support would have required some months of CPU time.)

2	x^2b^1	6	x^3b^2	18	x^4b^3
1	x^4b^4	4	$x^4b^3\lambda^1$	55	x^5b^4
8	x^5b^5	32	$x^5b^4\lambda^1$	174	x^6b^5
40	x^6b^6	2	x^6b^7	160	$x^6b^5\lambda^1$
14	$x^6b^6\lambda^1$	30	$x^6b^5\lambda^2$	570	x^7b^6
168	x^7b^7	22	x^7b^8	672	$x^7b^6\lambda^1$
156	$x^7b^7\lambda^1$	332	$x^7b^6\lambda^2$	1908	x^8b^7
677	x^8b^8	134	x^8b^9	6	x^8b^{10}
2712	$x^8b^7\lambda^1$	958	$x^8b^8\lambda^1$	60	$x^8b^9\lambda^1$
2030	$x^8b^7\lambda^2$	228	$x^8b^8\lambda^2$	336	$x^8b^7\lambda^3$
6473	x^9b^8	2708	x^9b^9	656	x^9b^{10}
72	x^9b^{11}	1	x^9b^{12}	10880	$x^9b^8\lambda^1$
4724	$x^9b^9\lambda^1$	728	$x^9b^{10}\lambda^1$	12	$x^9b^{11}\lambda^1$
9972	$x^9b^8\lambda^2$	2776	$x^9b^9\lambda^2$	62	$x^9b^{10}\lambda^2$
4064	$x^9b^8\lambda^3$	164	$x^9b^9\lambda^3$	192	$x^9b^8\lambda^4$
22202	$x^{10}b^9$	10724	$x^{10}b^{10}$	3008	$x^{10}b^{11}$
482	$x^{10}b^{12}$	30	$x^{10}b^{13}$	43220	$x^{10}b^9\lambda^1$
21844	$x^{10}b^{10}\lambda^1$	4920	$x^{10}b^{11}\lambda^1$	378	$x^{10}b^{12}\lambda^1$
46004	$x^{10}b^9\lambda^2$	18816	$x^{10}b^{10}\lambda^2$	2000	$x^{10}b^{11}\lambda^2$
27392	$x^{10}b^9\lambda^3$	5308	$x^{10}b^{10}\lambda^3$	6062	$x^{10}b^9\lambda^4$
76886	$x^{11}b^{10}$	42012	$x^{11}b^{11}$	13456	$x^{11}b^{12}$
2596	$x^{11}b^{13}$	310	$x^{11}b^{14}$	8	$x^{11}b^{15}$
169784	$x^{11}b^{10}\lambda^1$	98596	$x^{11}b^{11}\lambda^1$	26756	$x^{11}b^{12}\lambda^1$
3980	$x^{11}b^{13}\lambda^1$	120	$x^{11}b^{14}\lambda^1$	207444	$x^{11}b^{10}\lambda^2$
102660	$x^{11}b^{11}\lambda^2$	21268	$x^{11}b^{12}\lambda^2$	792	$x^{11}b^{13}\lambda^2$
148728	$x^{11}b^{10}\lambda^3$	56496	$x^{11}b^{11}\lambda^3$	2912	$x^{11}b^{12}\lambda^3$
63852	$x^{11}b^{10}\lambda^4$	6032	$x^{11}b^{11}\lambda^4$	5696	$x^{11}b^{10}\lambda^5$
268352	$x^{12}b^{11}$	163494	$x^{12}b^{12}$	58742	$x^{12}b^{13}$
13034	$x^{12}b^{14}$	2086	$x^{12}b^{15}$	151	$x^{12}b^{16}$
2	$x^{12}b^{17}$	662424	$x^{12}b^{11}\lambda^1$	433922	$x^{12}b^{12}\lambda^1$
135796	$x^{12}b^{13}\lambda^1$	27134	$x^{12}b^{14}\lambda^1$	2320	$x^{12}b^{15}\lambda^1$
34	$x^{12}b^{16}\lambda^1$	912378	$x^{12}b^{11}\lambda^2$	523436	$x^{12}b^{12}\lambda^2$
146048	$x^{12}b^{13}\lambda^2$	15582	$x^{12}b^{14}\lambda^2$	264	$x^{12}b^{15}\lambda^2$
755936	$x^{12}b^{11}\lambda^3$	388148	$x^{12}b^{12}\lambda^3$	57772	$x^{12}b^{13}\lambda^3$
1206	$x^{12}b^{14}\lambda^3$	435330	$x^{12}b^{11}\lambda^4$	119378	$x^{12}b^{12}\lambda^4$
3452	$x^{12}b^{13}\lambda^4$	111112	$x^{12}b^{11}\lambda^5$	5926	$x^{12}b^{12}\lambda^5$
4830	$x^{12}b^{11}\lambda^6$	942651	$x^{13}b^{12}$	633748	$x^{13}b^{13}$
250986	$x^{13}b^{14}$	63256	$x^{13}b^{15}$	11789	$x^{13}b^{16}$
1392	$x^{13}b^{17}$	68	$x^{13}b^{18}$	2573976	$x^{13}b^{12}\lambda^1$
1867280	$x^{13}b^{13}\lambda^1$	666112	$x^{13}b^{14}\lambda^1$	155240	$x^{13}b^{15}\lambda^1$
21744	$x^{13}b^{16}\lambda^1$	1196	$x^{13}b^{17}\lambda^1$	3923948	$x^{13}b^{12}\lambda^2$
2580304	$x^{13}b^{13}\lambda^2$	841656	$x^{13}b^{14}\lambda^2$	147808	$x^{13}b^{15}\lambda^2$
9508	$x^{13}b^{16}\lambda^2$	3718712	$x^{13}b^{12}\lambda^3$	2239824	$x^{13}b^{13}\lambda^3$
551136	$x^{13}b^{14}\lambda^3$	44080	$x^{13}b^{15}\lambda^3$	2497462	$x^{13}b^{12}\lambda^4$
1136672	$x^{13}b^{13}\lambda^4$	126872	$x^{13}b^{14}\lambda^4$	1047168	$x^{13}b^{12}\lambda^5$
216784	$x^{13}b^{13}\lambda^5$	173400	$x^{13}b^{12}\lambda^6$	3329608	$x^{14}b^{13}$
2448760	$x^{14}b^{14}$	1056608	$x^{14}b^{15}$	297262	$x^{14}b^{16}$
62396	$x^{14}b^{17}$	9354	$x^{14}b^{18}$	864	$x^{14}b^{19}$
22	$x^{14}b^{20}$	9967932	$x^{14}b^{13}\lambda^1$	7912384	$x^{14}b^{14}\lambda^1$
3160016	$x^{14}b^{15}\lambda^1$	832154	$x^{14}b^{16}\lambda^1$	148132	$x^{14}b^{17}\lambda^1$

15516	$x^{14}b^{18}\lambda^{1}$	440	$x^{14}b^{19}\lambda^{1}$	16621216	$x^{14}b^{13}\lambda^{2}$
12293820	$x^{14}b^{14}\lambda^{2}$	4547850	$x^{14}b^{15}\lambda^{2}$	1017002	$x^{14}b^{16}\lambda^{2}$
125358	$x^{14}b^{17}\lambda^{2}$	4044	$x^{14}b^{18}\lambda^{2}$	17685192	$x^{14}b^{13}\lambda^{3}$
12133344	$x^{14}b^{14}\lambda^{3}$	3810508	$x^{14}b^{15}\lambda^{3}$	587372	$x^{14}b^{16}\lambda^{3}$
22320	$x^{14}b^{17}\lambda^{3}$	13472960	$x^{14}b^{13}\lambda^{4}$	7848990	$x^{14}b^{14}\lambda^{4}$
1696946	$x^{14}b^{15}\lambda^{4}$	80572	$x^{14}b^{16}\lambda^{4}$	7173256	$x^{14}b^{13}\lambda^{5}$
2887776	$x^{14}b^{14}\lambda^{5}$	191816	$x^{14}b^{15}\lambda^{5}$	2280164	$x^{14}b^{13}\lambda^{6}$
280928	$x^{14}b^{14}\lambda^{6}$	196608	$x^{14}b^{13}\lambda^{7}$	11817582	$x^{15}b^{14}$
9436252	$x^{15}b^{15}$	4401192	$x^{15}b^{16}$	1359512	$x^{15}b^{17}$
317722	$x^{15}b^{18}$	54908	$x^{15}b^{19}$	7036	$x^{15}b^{20}$
456	$x^{15}b^{21}$	6	$x^{15}b^{22}$	38489344	$x^{15}b^{14}\lambda^{1}$
33157672	$x^{15}b^{15}\lambda^{1}$	14573424	$x^{15}b^{16}\lambda^{1}$	4288952	$x^{15}b^{17}\lambda^{1}$
881028	$x^{15}b^{18}\lambda^{1}$	128476	$x^{15}b^{19}\lambda^{1}$	9280	$x^{15}b^{20}\lambda^{1}$
132	$x^{15}b^{21}\lambda^{1}$	69641568	$x^{15}b^{14}\lambda^{2}$	56910344	$x^{15}b^{15}\lambda^{2}$
23620516	$x^{15}b^{16}\lambda^{2}$	6107840	$x^{15}b^{17}\lambda^{2}$	1051836	$x^{15}b^{18}\lambda^{2}$
86544	$x^{15}b^{19}\lambda^{2}$	1358	$x^{15}b^{20}\lambda^{2}$	81730120	$x^{15}b^{14}\lambda^{3}$
63187272	$x^{15}b^{15}\lambda^{3}$	23007296	$x^{15}b^{16}\lambda^{3}$	4971724	$x^{15}b^{17}\lambda^{3}$
482944	$x^{15}b^{18}\lambda^{3}$	8576	$x^{15}b^{19}\lambda^{3}$	69928992	$x^{15}b^{14}\lambda^{4}$
47396172	$x^{15}b^{15}\lambda^{4}$	14407296	$x^{15}b^{16}\lambda^{4}$	1754848	$x^{15}b^{17}\lambda^{4}$
36612	$x^{15}b^{18}\lambda^{4}$	43064560	$x^{15}b^{14}\lambda^{5}$	24432004	$x^{15}b^{15}\lambda^{5}$
4182976	$x^{15}b^{16}\lambda^{5}$	109012	$x^{15}b^{17}\lambda^{5}$	19087660	$x^{15}b^{14}\lambda^{6}$
6096976	$x^{15}b^{15}\lambda^{6}$	222732	$x^{15}b^{16}\lambda^{6}$	4218176	$x^{15}b^{14}\lambda^{7}$
287432	$x^{15}b^{15}\lambda^{7}$	180674	$x^{15}b^{14}\lambda^{8}$	42120340	$x^{16}b^{15}$
36285432	$x^{16}b^{16}$	18173796	$x^{16}b^{17}$	6095764	$x^{16}b^{18}$
1563218	$x^{16}b^{19}$	303068	$x^{16}b^{20}$	46352	$x^{16}b^{21}$
4748	$x^{16}b^{22}$	218	$x^{16}b^{23}$	1	$x^{16}b^{24}$
148278528	$x^{16}b^{15}\lambda^{1}$	137676732	$x^{16}b^{16}\lambda^{1}$	65843728	$x^{16}b^{17}\lambda^{1}$
21331470	$x^{16}b^{18}\lambda^{1}$	4926104	$x^{16}b^{19}\lambda^{1}$	859088	$x^{16}b^{20}\lambda^{1}$
98128	$x^{16}b^{21}\lambda^{1}$	4926	$x^{16}b^{22}\lambda^{1}$	24	$x^{16}b^{23}\lambda^{1}$
289148184	$x^{16}b^{15}\lambda^{2}$	258018196	$x^{16}b^{16}\lambda^{2}$	118276894	$x^{16}b^{17}\lambda^{2}$
34483370	$x^{16}b^{18}\lambda^{2}$	7117994	$x^{16}b^{19}\lambda^{2}$	927390	$x^{16}b^{20}\lambda^{2}$
51774	$x^{16}b^{21}\lambda^{2}$	272	$x^{16}b^{22}\lambda^{2}$	370021256	$x^{16}b^{15}\lambda^{3}$
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5228216	$x^{16}b^{19}\lambda^{3}$	332474	$x^{16}b^{20}\lambda^{3}$	1920	$x^{16}b^{21}\lambda^{3}$
349897084	$x^{16}b^{15}\lambda^{4}$	269377798	$x^{16}b^{16}\lambda^{4}$	98616890	$x^{16}b^{17}\lambda^{4}$
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243682636	$x^{16}b^{15}\lambda^{5}$	166869768	$x^{16}b^{16}\lambda^{5}$	45604660	$x^{16}b^{17}\lambda^{5}$
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45287440	$x^{16}b^{15}\lambda^{7}$	11259232	$x^{16}b^{16}\lambda^{7}$	151160	$x^{16}b^{17}\lambda^{7}$
6961342	$x^{16}b^{15}\lambda^{8}$	175264	$x^{16}b^{16}\lambda^{8}$	100352	$x^{16}b^{15}\lambda^{9}$
150682450	$x^{17}b^{16}$	139297108	$x^{17}b^{17}$	74496544	$x^{17}b^{18}$
26922156	$x^{17}b^{19}$	7477928	$x^{17}b^{20}$	1603984	$x^{17}b^{21}$
276464	$x^{17}b^{22}$	36112	$x^{17}b^{23}$	3010	$x^{17}b^{24}$
88	$x^{17}b^{25}$	570197440	$x^{17}b^{16}\lambda^{1}$	567215448	$x^{17}b^{17}\lambda^{1}$
292841664	$x^{17}b^{18}\lambda^{1}$	103037792	$x^{17}b^{19}\lambda^{1}$	26386768	$x^{17}b^{20}\lambda^{1}$
5197132	$x^{17}b^{21}\lambda^{1}$	757456	$x^{17}b^{22}\lambda^{1}$	69140	$x^{17}b^{23}\lambda^{1}$
2184	$x^{17}b^{24}\lambda^{1}$	1191271268	$x^{17}b^{16}\lambda^{2}$	1151253208	$x^{17}b^{17}\lambda^{2}$
574774622	$x^{17}b^{18}\lambda^{2}$	186372724	$x^{17}b^{19}\lambda^{2}$	43556888	$x^{17}b^{20}\lambda^{2}$
7250888	$x^{17}b^{21}\lambda^{2}$	737000	$x^{17}b^{22}\lambda^{2}$	25464	$x^{17}b^{23}\lambda^{2}$
1649049272	$x^{17}b^{16}\lambda^{3}$	1544225612	$x^{17}b^{17}\lambda^{3}$	709770552	$x^{17}b^{18}\lambda^{3}$
209234208	$x^{17}b^{19}\lambda^{3}$	41283508	$x^{17}b^{20}\lambda^{3}$	4787332	$x^{17}b^{21}\lambda^{3}$
184000	$x^{17}b^{22}\lambda^{3}$	1699165082	$x^{17}b^{16}\lambda^{4}$	1465060796	$x^{17}b^{17}\lambda^{4}$
610734500	$x^{17}b^{18}\lambda^{4}$	151832356	$x^{17}b^{19}\lambda^{4}$	20851142	$x^{17}b^{20}\lambda^{4}$
913352	$x^{17}b^{21}\lambda^{4}$	1320197000	$x^{17}b^{16}\lambda^{5}$	1032598112	$x^{17}b^{17}\lambda^{5}$
362746612	$x^{17}b^{18}\lambda^{5}$	62739052	$x^{17}b^{19}\lambda^{5}$	3250192	$x^{17}b^{20}\lambda^{5}$

795256592	$x^{17}b^{16}\lambda^{6}$	524151628	$x^{17}b^{17}\lambda^{6}$	128110568	$x^{17}b^{18}\lambda^{6}$
8344176	$x^{17}b^{19}\lambda^{6}$	355393128	$x^{17}b^{16}\lambda^{7}$	163177976	$x^{17}b^{17}\lambda^{7}$
14991056	$x^{17}b^{18}\lambda^{7}$	99763798	$x^{17}b^{16}\lambda^{8}$	17190168	$x^{17}b^{17}\lambda^{8}$
9630560	$x^{17}b^{16}\lambda^{9}$	540832274	$x^{18}b^{17}$	534018776	$x^{18}b^{18}$
303516966	$x^{18}b^{19}$	117417380	$x^{18}b^{20}$	35016382	$x^{18}b^{21}$
8193956	$x^{18}b^{22}$	1559888	$x^{18}b^{23}$	236256	$x^{18}b^{24}$
26906	$x^{18}b^{25}$	1728	$x^{18}b^{26}$	30	$x^{18}b^{27}$
2189348656	$x^{18}b^{17}\lambda^{1}$	2321743284	$x^{18}b^{18}\lambda^{1}$	1285388800	$x^{18}b^{19}\lambda^{1}$
486782762	$x^{18}b^{20}\lambda^{1}$	136269988	$x^{18}b^{21}\lambda^{1}$	29719152	$x^{18}b^{22}\lambda^{1}$
5027456	$x^{18}b^{23}\lambda^{1}$	627126	$x^{18}b^{24}\lambda^{1}$	43640	$x^{18}b^{25}\lambda^{1}$
810	$x^{18}b^{26}\lambda^{1}$	4876431544	$x^{18}b^{17}\lambda^{2}$	5068308128	$x^{18}b^{18}\lambda^{2}$
2730455738	$x^{18}b^{19}\lambda^{2}$	970230636	$x^{18}b^{20}\lambda^{2}$	251776888	$x^{18}b^{21}\lambda^{2}$
48730120	$x^{18}b^{22}\lambda^{2}$	6772030	$x^{18}b^{23}\lambda^{2}$	516676	$x^{18}b^{24}\lambda^{2}$
10358	$x^{18}b^{25}\lambda^{2}$	7252368728	$x^{18}b^{17}\lambda^{3}$	7349733768	$x^{18}b^{18}\lambda^{3}$
3712183136	$x^{18}b^{19}\lambda^{3}$	1218763220	$x^{18}b^{20}\lambda^{3}$	280171364	$x^{18}b^{21}\lambda^{3}$
44463194	$x^{18}b^{22}\lambda^{3}$	3782312	$x^{18}b^{23}\lambda^{3}$	82952	$x^{18}b^{24}\lambda^{3}$
8068090292	$x^{18}b^{17}\lambda^{4}$	7668659064	$x^{18}b^{18}\lambda^{4}$	3571556628	$x^{18}b^{19}\lambda^{4}$
1036926064	$x^{18}b^{20}\lambda^{4}$	195165630	$x^{18}b^{21}\lambda^{4}$	18968628	$x^{18}b^{22}\lambda^{4}$
463000	$x^{18}b^{23}\lambda^{4}$	6885986552	$x^{18}b^{17}\lambda^{5}$	6037033596	$x^{18}b^{18}\lambda^{5}$
2482681364	$x^{18}b^{19}\lambda^{5}$	589603862	$x^{18}b^{20}\lambda^{5}$	67970592	$x^{18}b^{21}\lambda^{5}$
1892332	$x^{18}b^{22}\lambda^{5}$	4626204916	$x^{18}b^{17}\lambda^{6}$	3578361724	$x^{18}b^{18}\lambda^{6}$
1203519470	$x^{18}b^{19}\lambda^{6}$	175017280	$x^{18}b^{20}\lambda^{6}$	5768984	$x^{18}b^{21}\lambda^{6}$
2408178984	$x^{18}b^{17}\lambda^{7}$	1525073160	$x^{18}b^{18}\lambda^{7}$	313925560	$x^{18}b^{19}\lambda^{7}$
13028288	$x^{18}b^{20}\lambda^{7}$	922980400	$x^{18}b^{17}\lambda^{8}$	357584344	$x^{18}b^{18}\lambda^{8}$
20968600	$x^{18}b^{19}\lambda^{8}$	197901280	$x^{18}b^{17}\lambda^{9}$	21816740	$x^{18}b^{18}\lambda^{9}$
11189428	$x^{18}b^{17}\lambda^{10}$	1946892842	$x^{19}b^{18}$	2044899048	$x^{19}b^{19}$
1230292398	$x^{19}b^{20}$	506666828	$x^{19}b^{21}$	161217996	$x^{19}b^{22}$
40680552	$x^{19}b^{23}$	8446952	$x^{19}b^{24}$	1427768	$x^{19}b^{25}$
194614	$x^{19}b^{26}$	18756	$x^{19}b^{27}$	914	$x^{19}b^{28}$
8	$x^{19}b^{29}$	8395558272	$x^{19}b^{18}\lambda^{1}$	9451518124	$x^{19}b^{19}\lambda^{1}$
5579213064	$x^{19}b^{20}\lambda^{1}$	2259475952	$x^{19}b^{21}\lambda^{1}$	683255864	$x^{19}b^{22}\lambda^{1}$
162928064	$x^{19}b^{23}\lambda^{1}$	30804952	$x^{19}b^{24}\lambda^{1}$	4602032	$x^{19}b^{25}\lambda^{1}$
480468	$x^{19}b^{26}\lambda^{1}$	25132	$x^{19}b^{27}\lambda^{1}$	232	$x^{19}b^{28}\lambda^{1}$
19853269060	$x^{19}b^{18}\lambda^{2}$	22059187448	$x^{19}b^{19}\lambda^{2}$	12738290144	$x^{19}b^{20}\lambda^{2}$
4900067724	$x^{19}b^{21}\lambda^{2}$	1393960008	$x^{19}b^{22}\lambda^{2}$	302154544	$x^{19}b^{23}\lambda^{2}$
50341268	$x^{19}b^{24}\lambda^{2}$	5762384	$x^{19}b^{25}\lambda^{2}$	326576	$x^{19}b^{26}\lambda^{2}$
3208	$x^{19}b^{27}\lambda^{2}$	31536512904	$x^{19}b^{18}\lambda^{3}$	34349345640	$x^{19}b^{19}\lambda^{3}$
18826752584	$x^{19}b^{20}\lambda^{3}$	6794650376	$x^{19}b^{21}\lambda^{3}$	1753568616	$x^{19}b^{22}\lambda^{3}$
334116304	$x^{19}b^{23}\lambda^{3}$	42656552	$x^{19}b^{24}\lambda^{3}$	2651928	$x^{19}b^{25}\lambda^{3}$
28016	$x^{19}b^{26}\lambda^{3}$	37628498868	$x^{19}b^{18}\lambda^{4}$	38921156884	$x^{19}b^{19}\lambda^{4}$
19982584248	$x^{19}b^{20}\lambda^{4}$	6530827336	$x^{19}b^{21}\lambda^{4}$	1478341424	$x^{19}b^{22}\lambda^{4}$
215834112	$x^{19}b^{23}\lambda^{4}$	14973320	$x^{19}b^{24}\lambda^{4}$	172416	$x^{19}b^{25}\lambda^{4}$
34838434432	$x^{19}b^{18}\lambda^{5}$	33768418840	$x^{19}b^{19}\lambda^{5}$	15677650064	$x^{19}b^{20}\lambda^{5}$
4486332184	$x^{19}b^{21}\lambda^{5}$	778048216	$x^{19}b^{22}\lambda^{5}$	61737300	$x^{19}b^{23}\lambda^{5}$
788952	$x^{19}b^{24}\lambda^{5}$	25760261240	$x^{19}b^{18}\lambda^{6}$	22563569224	$x^{19}b^{19}\lambda^{6}$
9161603600	$x^{19}b^{20}\lambda^{6}$	2008379728	$x^{19}b^{21}\lambda^{6}$	189255452	$x^{19}b^{22}\lambda^{6}$
2753800	$x^{19}b^{23}\lambda^{6}$	15095548080	$x^{19}b^{18}\lambda^{7}$	11562493880	$x^{19}b^{19}\lambda^{7}$
3596873176	$x^{19}b^{20}\lambda^{7}$	428131600	$x^{19}b^{21}\lambda^{7}$	7373072	$x^{19}b^{22}\lambda^{7}$
6936931892	$x^{19}b^{18}\lambda^{8}$	4073064444	$x^{19}b^{19}\lambda^{8}$	687254816	$x^{19}b^{20}\lambda^{8}$
14920032	$x^{19}b^{21}\lambda^{8}$	2230866664	$x^{19}b^{18}\lambda^{9}$	709772620	$x^{19}b^{19}\lambda^{9}$
21837608	$x^{19}b^{20}\lambda^{9}$	359409332	$x^{19}b^{18}\lambda^{10}$	20891784	$x^{19}b^{19}\lambda^{10}$
9934752	$x^{19}b^{18}\lambda^{11}$				

Appendix B. The results for the simple cubic lattice are given below; see appendix A for the explanation of the layout. These results required nearly 7 hours of CRAY CPU time. A direct count would have taken a few years to complete.

3	x^2b^1	15	x^3b^2	83	x^4b^3
3	x^4b^4	12	$x^4b^3\lambda^1$	486	x^5b^4
48	x^5b^5	192	$x^5b^4\lambda^1$	2967	x^6b^5
496	x^6b^6	18	x^6b^7	1992	$x^6b^5\lambda^1$
126	$x^6b^6\lambda^1$	270	$x^6b^5\lambda^2$	18748	x^7b^6
4368	x^7b^7	378	x^7b^8	8	x^7b^9
17616	$x^7b^6\lambda^1$	2676	$x^7b^7\lambda^1$	72	$x^7b^8\lambda^1$
5700	$x^7b^6\lambda^2$	264	$x^7b^7\lambda^2$	400	$x^7b^6\lambda^3$
121725	x^8b^7	36027	x^8b^8	4854	x^8b^9
306	x^8b^{10}	1	x^8b^{12}	145872	$x^8b^7\lambda^1$
34782	$x^8b^8\lambda^1$	2892	$x^8b^9\lambda^1$	12	$x^8b^{11}\lambda^1$
73902	$x^8b^7\lambda^2$	10764	$x^8b^8\lambda^2$	66	$x^8b^{10}\lambda^2$
16104	$x^8b^7\lambda^3$	212	$x^8b^9\lambda^3$	408	$x^8b^8\lambda^4$
384	$x^8b^7\lambda^5$	807381	x^9b^8	288732	x^9b^9
51030	x^9b^{10}	5544	x^9b^{11}	159	x^9b^{12}
24	x^9b^{13}	1173216	$x^9b^8\lambda^1$	370032	$x^9b^9\lambda^1$
53736	$x^9b^{10}\lambda^1$	1908	$x^9b^{11}\lambda^1$	288	$x^9b^{12}\lambda^1$
785448	$x^9b^8\lambda^2$	201768	$x^9b^9\lambda^2$	9666	$x^9b^{10}\lambda^2$
1584	$x^9b^{11}\lambda^2$	299472	$x^9b^8\lambda^3$	25212	$x^9b^9\lambda^3$
5088	$x^9b^{10}\lambda^3$	29280	$x^9b^8\lambda^4$	9792	$x^9b^9\lambda^4$
9216	$x^9b^8\lambda^5$	5447203	$x^{10}b^9$	2280792	$x^{10}b^{10}$
488976	$x^{10}b^{11}$	72244	$x^{10}b^{12}$	5103	$x^{10}b^{13}$
396	$x^{10}b^{14}$	24	$x^{10}b^{15}$	9296964	$x^{10}b^9\lambda^1$
3585744	$x^{10}b^{10}\lambda^1$	713568	$x^{10}b^{11}\lambda^1$	62553	$x^{10}b^{12}\lambda^1$
4896	$x^{10}b^{13}\lambda^1$	360	$x^{10}b^{14}\lambda^1$	7608912	$x^{10}b^9\lambda^2$
2699952	$x^{10}b^{10}\lambda^2$	321738	$x^{10}b^{11}\lambda^2$	27660	$x^{10}b^{12}\lambda^2$
2448	$x^{10}b^{13}\lambda^2$	3986592	$x^{10}b^9\lambda^3$	843138	$x^{10}b^{10}\lambda^3$
90744	$x^{10}b^{11}\lambda^3$	9888	$x^{10}b^{12}\lambda^3$	970845	$x^{10}b^9\lambda^4$
177060	$x^{10}b^{10}\lambda^4$	25608	$x^{10}b^{11}\lambda^4$	167760	$x^{10}b^9\lambda^5$
41280	$x^{10}b^{10}\lambda^5$	33024	$x^{10}b^9\lambda^6$	37264974	$x^{11}b^{10}$
17866896	$x^{11}b^{11}$	4463316	$x^{11}b^{12}$	801396	$x^{11}b^{13}$
89715	$x^{11}b^{14}$	7568	$x^{11}b^{15}$	660	$x^{11}b^{16}$
24	$x^{11}b^{17}$	73034952	$x^{11}b^{10}\lambda^1$	33069168	$x^{11}b^{11}\lambda^1$
8047308	$x^{11}b^{12}\lambda^1$	1118586	$x^{11}b^{13}\lambda^1$	101520	$x^{11}b^{14}\lambda^1$
9936	$x^{11}b^{15}\lambda^1$	408	$x^{11}b^{16}\lambda^1$	70171248	$x^{11}b^{10}\lambda^2$
30675228	$x^{11}b^{11}\lambda^2$	5823444	$x^{11}b^{12}\lambda^2$	613464	$x^{11}b^{13}\lambda^2$
67800	$x^{11}b^{14}\lambda^2$	3216	$x^{11}b^{15}\lambda^2$	45126408	$x^{11}b^{10}\lambda^3$
15323652	$x^{11}b^{11}\lambda^3$	2112640	$x^{11}b^{12}\lambda^3$	274728	$x^{11}b^{13}\lambda^3$
15384	$x^{11}b^{14}\lambda^3$	17533428	$x^{11}b^{10}\lambda^4$	4226472	$x^{11}b^{11}\lambda^4$
713256	$x^{11}b^{12}\lambda^4$	49008	$x^{11}b^{13}\lambda^4$	4004592	$x^{11}b^{10}\lambda^5$
1151064	$x^{11}b^{11}\lambda^5$	106104	$x^{11}b^{12}\lambda^5$	919680	$x^{11}b^{10}\lambda^6$
148008	$x^{11}b^{11}\lambda^6$	104880	$x^{11}b^{10}\lambda^7$	257896500	$x^{12}b^{11}$
139239286	$x^{12}b^{12}$	39546852	$x^{12}b^{13}$	8179476	$x^{12}b^{14}$
1197481	$x^{12}b^{15}$	132681	$x^{12}b^{16}$	12546	$x^{12}b^{17}$
1080	$x^{12}b^{18}$	3	$x^{12}b^{20}$	570616752	$x^{12}b^{11}\lambda^1$
295734240	$x^{12}b^{12}\lambda^1$	83392416	$x^{12}b^{13}\lambda^1$	15160359	$x^{12}b^{14}\lambda^1$
1883760	$x^{12}b^{15}\lambda^1$	194130	$x^{12}b^{16}\lambda^1$	18744	$x^{12}b^{17}\lambda^1$
60	$x^{12}b^{19}\lambda^1$	627603288	$x^{12}b^{11}\lambda^2$	320164692	$x^{12}b^{12}\lambda^2$
79791720	$x^{12}b^{13}\lambda^2$	11886678	$x^{12}b^{14}\lambda^2$	1362480	$x^{12}b^{15}\lambda^2$
150000	$x^{12}b^{16}\lambda^2$	570	$x^{12}b^{18}\lambda^2$	469676808	$x^{12}b^{11}\lambda^3$
210815986	$x^{12}b^{12}\lambda^3$	42137556	$x^{12}b^{13}\lambda^3$	5665050	$x^{12}b^{14}\lambda^3$
725688	$x^{12}b^{15}\lambda^3$	3396	$x^{12}b^{17}\lambda^3$	240021897	$x^{12}b^{11}\lambda^4$
85344114	$x^{12}b^{12}\lambda^4$	15011232	$x^{12}b^{13}\lambda^4$	2330616	$x^{12}b^{14}\lambda^4$

14085	$x^{12}b^{16}\lambda^{4}$	80393760	$x^{12}b^{11}\lambda^{5}$	24531618	$x^{12}b^{12}\lambda^{5}$
5068080	$x^{12}b^{13}\lambda^{5}$	42576	$x^{12}b^{15}\lambda^{5}$	19664922	$x^{12}b^{11}\lambda^{6}$
7060968	$x^{12}b^{12}\lambda^{6}$	95058	$x^{12}b^{14}\lambda^{6}$	4958880	$x^{12}b^{11}\lambda^{7}$
153828	$x^{12}b^{13}\lambda^{7}$	167055	$x^{12}b^{12}\lambda^{8}$	94500	$x^{12}b^{11}\lambda^{9}$
1802312605	$x^{13}b^{12}$	1081555008	$x^{13}b^{13}$	343288410	$x^{13}b^{14}$
79574192	$x^{13}b^{15}$	13869918	$x^{13}b^{16}$	1929048	$x^{13}b^{17}$
215204	$x^{13}b^{18}$	23976	$x^{13}b^{19}$	864	$x^{13}b^{20}$
96	$x^{13}b^{21}$	4442485104	$x^{13}b^{12}\lambda^{1}$	2588452080	$x^{13}b^{13}\lambda^{1}$
822711240	$x^{13}b^{14}\lambda^{1}$	178172184	$x^{13}b^{15}\lambda^{1}$	28331448	$x^{13}b^{16}\lambda^{1}$
3469692	$x^{13}b^{17}\lambda^{1}$	422856	$x^{13}b^{18}\lambda^{1}$	17280	$x^{13}b^{19}\lambda^{1}$
1920	$x^{13}b^{20}\lambda^{1}$	5494079484	$x^{13}b^{12}\lambda^{2}$	3179969448	$x^{13}b^{13}\lambda^{2}$
947552586	$x^{13}b^{14}\lambda^{2}$	183373536	$x^{13}b^{15}\lambda^{2}$	25344360	$x^{13}b^{16}\lambda^{2}$
3431400	$x^{13}b^{17}\lambda^{2}$	160368	$x^{13}b^{18}\lambda^{2}$	18240	$x^{13}b^{19}\lambda^{2}$
4654566416	$x^{13}b^{12}\lambda^{3}$	2513815812	$x^{13}b^{13}\lambda^{3}$	660932832	$x^{13}b^{14}\lambda^{3}$
109090952	$x^{13}b^{15}\lambda^{3}$	16806864	$x^{13}b^{16}\lambda^{3}$	910344	$x^{13}b^{17}\lambda^{3}$
108672	$x^{13}b^{18}\lambda^{3}$	2850265746	$x^{13}b^{12}\lambda^{4}$	1346946696	$x^{13}b^{13}\lambda^{4}$
296534484	$x^{13}b^{14}\lambda^{4}$	54544752	$x^{13}b^{15}\lambda^{4}$	3497736	$x^{13}b^{16}\lambda^{4}$
450720	$x^{13}b^{17}\lambda^{4}$	1261429248	$x^{13}b^{12}\lambda^{5}$	491026992	$x^{13}b^{13}\lambda^{5}$
119519664	$x^{13}b^{14}\lambda^{5}$	9453384	$x^{13}b^{15}\lambda^{5}$	1362432	$x^{13}b^{16}\lambda^{5}$
393237032	$x^{13}b^{12}\lambda^{6}$	167057664	$x^{13}b^{13}\lambda^{6}$	17859576	$x^{13}b^{14}\lambda^{6}$
3041856	$x^{13}b^{15}\lambda^{6}$	117012768	$x^{13}b^{12}\lambda^{7}$	21973632	$x^{13}b^{13}\lambda^{7}$
4922496	$x^{13}b^{14}\lambda^{7}$	13714224	$x^{13}b^{12}\lambda^{8}$	5345760	$x^{13}b^{13}\lambda^{8}$
3024000	$x^{13}b^{12}\lambda^{9}$	12701190885	$x^{14}b^{13}$	8383668876	$x^{14}b^{14}$
2935371280	$x^{14}b^{15}$	750063441	$x^{14}b^{16}$	148459755	$x^{14}b^{17}$
24310222	$x^{14}b^{18}$	3261084	$x^{14}b^{19}$	412998	$x^{14}b^{20}$
32116	$x^{14}b^{21}$	2340	$x^{14}b^{22}$	114	$x^{14}b^{23}$
34507622736	$x^{14}b^{13}\lambda^{1}$	22297115568	$x^{14}b^{14}\lambda^{1}$	7854272820	$x^{14}b^{15}\lambda^{1}$
1934055645	$x^{14}b^{16}\lambda^{1}$	365670468	$x^{14}b^{17}\lambda^{1}$	54433752	$x^{14}b^{18}\lambda^{1}$
7468344	$x^{14}b^{19}\lambda^{1}$	648684	$x^{14}b^{20}\lambda^{1}$	48192	$x^{14}b^{21}\lambda^{1}$
2622	$x^{14}b^{22}\lambda^{1}$	47335340712	$x^{14}b^{13}\lambda^{2}$	30547256808	$x^{14}b^{14}\lambda^{2}$
10388942088	$x^{14}b^{15}\lambda^{2}$	2410082202	$x^{14}b^{16}\lambda^{2}$	410528778	$x^{14}b^{17}\lambda^{2}$
62091204	$x^{14}b^{18}\lambda^{2}$	6072840	$x^{14}b^{19}\lambda^{2}$	470784	$x^{14}b^{20}\lambda^{2}$
28518	$x^{14}b^{21}\lambda^{2}$	44629965192	$x^{14}b^{13}\lambda^{3}$	27674463246	$x^{14}b^{14}\lambda^{3}$
8790234092	$x^{14}b^{15}\lambda^{3}$	1814255004	$x^{14}b^{16}\lambda^{3}$	311193024	$x^{14}b^{17}\lambda^{3}$
34736116	$x^{14}b^{18}\lambda^{3}$	2878056	$x^{14}b^{19}\lambda^{3}$	194514	$x^{14}b^{20}\lambda^{3}$
31267320963	$x^{14}b^{13}\lambda^{4}$	17988227118	$x^{14}b^{14}\lambda^{4}$	5022483090	$x^{14}b^{15}\lambda^{4}$
1030558956	$x^{14}b^{16}\lambda^{4}$	134333280	$x^{14}b^{17}\lambda^{4}$	12217248	$x^{14}b^{18}\lambda^{4}$
928158	$x^{14}b^{19}\lambda^{4}$	16759746024	$x^{14}b^{13}\lambda^{5}$	8382480456	$x^{14}b^{14}\lambda^{5}$
2292850680	$x^{14}b^{15}\lambda^{5}$	364938948	$x^{14}b^{16}\lambda^{5}$	37694976	$x^{14}b^{17}\lambda^{5}$
3264354	$x^{14}b^{18}\lambda^{5}$	6685661748	$x^{14}b^{13}\lambda^{6}$	3229884000	$x^{14}b^{14}\lambda^{6}$
691651220	$x^{14}b^{15}\lambda^{6}$	85631400	$x^{14}b^{16}\lambda^{6}$	8656854	$x^{14}b^{17}\lambda^{6}$
2258748744	$x^{14}b^{13}\lambda^{7}$	851145936	$x^{14}b^{14}\lambda^{7}$	140420064	$x^{14}b^{15}\lambda^{7}$
17313048	$x^{14}b^{16}\lambda^{7}$	529155132	$x^{14}b^{13}\lambda^{8}$	153720552	$x^{14}b^{14}\lambda^{8}$
25380696	$x^{14}b^{15}\lambda^{8}$	87143832	$x^{14}b^{13}\lambda^{9}$	25049292	$x^{14}b^{14}\lambda^{9}$
12835236	$x^{14}b^{13}\lambda^{10}$	90157130289	$x^{15}b^{14}$	64901900852	$x^{15}b^{15}$
24813944160	$x^{15}b^{16}$	6909195930	$x^{15}b^{17}$	1513638408	$x^{15}b^{18}$
279429960	$x^{15}b^{19}$	43672152	$x^{15}b^{20}$	6239792	$x^{15}b^{21}$
684339	$x^{15}b^{22}$	60216	$x^{15}b^{23}$	4180	$x^{15}b^{24}$
168	$x^{15}b^{25}$	267647434752	$x^{15}b^{14}\lambda^{1}$	189736820532	$x^{15}b^{15}\lambda^{1}$
73187934300	$x^{15}b^{16}\lambda^{1}$	19984259004	$x^{15}b^{17}\lambda^{1}$	4285706520	$x^{15}b^{18}\lambda^{1}$
748657740	$x^{15}b^{19}\lambda^{1}$	116097768	$x^{15}b^{20}\lambda^{1}$	14017650	$x^{15}b^{21}\lambda^{1}$
1301064	$x^{15}b^{22}\lambda^{1}$	96612	$x^{15}b^{23}\lambda^{1}$	4200	$x^{15}b^{24}\lambda^{1}$
402881113224	$x^{15}b^{14}\lambda^{2}$	286248399972	$x^{15}b^{15}\lambda^{2}$	108368640174	$x^{15}b^{16}\lambda^{2}$
28672047480	$x^{15}b^{17}\lambda^{2}$	5782109052	$x^{15}b^{18}\lambda^{2}$	992272704	$x^{15}b^{19}\lambda^{2}$
133039173	$x^{15}b^{20}\lambda^{2}$	13263840	$x^{15}b^{21}\lambda^{2}$	1056108	$x^{15}b^{22}\lambda^{2}$
50088	$x^{15}b^{23}\lambda^{2}$	417554922912	$x^{15}b^{14}\lambda^{3}$	289819604188	$x^{15}b^{15}\lambda^{3}$

105606018600	$x^{15}b^{16}\lambda^{3}$	26044802448	$x^{15}b^{17}\lambda^{3}$	5102979896	$x^{15}b^{18}\lambda^{3}$
771071064	$x^{15}b^{19}\lambda^{3}$	84151224	$x^{15}b^{20}\lambda^{3}$	7240084	$x^{15}b^{21}\lambda^{3}$
377520	$x^{15}b^{22}\lambda^{3}$	326433287382	$x^{15}b^{14}\lambda^{4}$	216835734432	$x^{15}b^{15}\lambda^{4}$
73023125820	$x^{15}b^{16}\lambda^{4}$	17275219344	$x^{15}b^{17}\lambda^{4}$	3018843807	$x^{15}b^{18}\lambda^{4}$
368722488	$x^{15}b^{19}\lambda^{4}$	34719864	$x^{15}b^{20}\lambda^{4}$	2007960	$x^{15}b^{21}\lambda^{4}$
201109725768	$x^{15}b^{14}\lambda^{5}$	122444101176	$x^{15}b^{15}\lambda^{5}$	39049798872	$x^{15}b^{16}\lambda^{5}$
8289126774	$x^{15}b^{17}\lambda^{5}$	1168170216	$x^{15}b^{18}\lambda^{5}$	122697156	$x^{15}b^{19}\lambda^{5}$
7962264	$x^{15}b^{20}\lambda^{5}$	97191844656	$x^{15}b^{14}\lambda^{6}$	55408772232	$x^{15}b^{15}\lambda^{6}$
15834007176	$x^{15}b^{16}\lambda^{6}$	2709684264	$x^{15}b^{17}\lambda^{6}$	326855620	$x^{15}b^{18}\lambda^{6}$
24204024	$x^{15}b^{19}\lambda^{6}$	38621502576	$x^{15}b^{14}\lambda^{7}$	19556706000	$x^{15}b^{15}\lambda^{7}$
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59724096	$x^{15}b^{14}\lambda^{11}$	644022007040	$x^{16}b^{15}$	502042154466	$x^{16}b^{16}$
207901068732	$x^{16}b^{17}$	62526175719	$x^{16}b^{18}$	14925358275	$x^{16}b^{19}$
3027617862	$x^{16}b^{20}$	532143926	$x^{16}b^{21}$	85144794	$x^{16}b^{22}$
11390724	$x^{16}b^{23}$	1274990	$x^{16}b^{24}$	109830	$x^{16}b^{25}$
8262	$x^{16}b^{26}$	120	$x^{16}b^{27}$	15	$x^{16}b^{28}$
2073965899752	$x^{16}b^{15}\lambda^{1}$	1599173537226	$x^{16}b^{16}\lambda^{1}$	669256162884	$x^{16}b^{17}\lambda^{1}$
199545312735	$x^{16}b^{18}\lambda^{1}$	47234330016	$x^{16}b^{19}\lambda^{1}$	9322099746	$x^{16}b^{20}\lambda^{1}$
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2599446	$x^{16}b^{24}\lambda^{1}$	209052	$x^{16}b^{25}\lambda^{1}$	3240	$x^{16}b^{26}\lambda^{1}$
420	$x^{16}b^{27}\lambda^{1}$	3396362370270	$x^{16}b^{15}\lambda^{2}$	2630836381212	$x^{16}b^{16}\lambda^{2}$
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29097390	$x^{16}b^{23}\lambda^{2}$	2517582	$x^{16}b^{24}\lambda^{2}$	41856	$x^{16}b^{25}\lambda^{2}$
5670	$x^{16}b^{26}\lambda^{2}$	3832392373520	$x^{16}b^{15}\lambda^{3}$	2930400452790	$x^{16}b^{16}\lambda^{3}$
1189681304640	$x^{16}b^{17}\lambda^{3}$	335518826906	$x^{16}b^{18}\lambda^{3}$	74954699424	$x^{16}b^{19}\lambda^{3}$
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3291565921095	$x^{16}b^{15}\lambda^{4}$	2449940110662	$x^{16}b^{16}\lambda^{4}$	949948154034	$x^{16}b^{17}\lambda^{4}$
258518489430	$x^{16}b^{18}\lambda^{4}$	53951057676	$x^{16}b^{19}\lambda^{4}$	8647359684	$x^{16}b^{20}\lambda^{4}$
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303288	$x^{16}b^{24}\lambda^{4}$	2263013083236	$x^{16}b^{15}\lambda^{5}$	1597900554390	$x^{16}b^{16}\lambda^{5}$
591892581420	$x^{16}b^{17}\lambda^{5}$	150291200730	$x^{16}b^{18}\lambda^{5}$	27862262808	$x^{16}b^{19}\lambda^{5}$
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1425312	$x^{16}b^{23}\lambda^{5}$	1262410265762	$x^{16}b^{15}\lambda^{6}$	844040246052	$x^{16}b^{16}\lambda^{6}$
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19914886896	$x^{16}b^{18}\lambda^{7}$	2945819712	$x^{16}b^{19}\lambda^{7}$	80040288	$x^{16}b^{20}\lambda^{7}$
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1729547185260	$x^{17}b^{18}$	557941445832	$x^{17}b^{19}$	143556085446	$x^{17}b^{20}$
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164458571	$x^{17}b^{24}$	22048152	$x^{17}b^{25}$	2381526	$x^{17}b^{26}$

226464	$x^{17}b^{27}$	11694	$x^{17}b^{28}$	576	$x^{17}b^{29}$
24	$x^{17}b^{30}$	16061510248344	$x^{17}b^{16}\lambda^{1}$	13375407996960	$x^{17}b^{17}\lambda^{1}$
6028464278220	$x^{17}b^{18}\lambda^{1}$	1941807519936	$x^{17}b^{19}\lambda^{1}$	499181770776	$x^{17}b^{20}\lambda^{1}$
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323064	$x^{17}b^{27}\lambda^{1}$	16152	$x^{17}b^{28}\lambda^{1}$	720	$x^{17}b^{29}\lambda^{1}$
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71104464	$x^{17}b^{25}\lambda^{2}$	4260816	$x^{17}b^{26}\lambda^{2}$	218352	$x^{17}b^{27}\lambda^{2}$
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210941556	$x^{17}b^{24}\lambda^{4}$	11708760	$x^{17}b^{25}\lambda^{4}$	643152	$x^{17}b^{26}\lambda^{4}$
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37000465240	$x^{17}b^{21}\lambda^{6}$	3230726988	$x^{17}b^{22}\lambda^{6}$	203364672	$x^{17}b^{23}\lambda^{6}$
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8749605768	$x^{17}b^{21}\lambda^{7}$	599747472	$x^{17}b^{22}\lambda^{7}$	42336000	$x^{17}b^{23}\lambda^{7}$
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111546864	$x^{17}b^{22}\lambda^{8}$	1320293720400	$x^{17}b^{16}\lambda^{9}$	749108389272	$x^{17}b^{17}\lambda^{9}$
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584368224	$x^{17}b^{19}\lambda^{11}$	14202593592	$x^{17}b^{16}\lambda^{12}$	3577997928	$x^{17}b^{17}\lambda^{12}$
620607024	$x^{17}b^{18}\lambda^{12}$	1465427712	$x^{17}b^{16}\lambda^{13}$	456732048	$x^{17}b^{17}\lambda^{13}$
178124544	$x^{17}b^{16}\lambda^{14}$				

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